GREEK METHODS OF SOLVING QUADRATIC EQUATIONS.

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Principal Greek mathematicians referred to in this paper with the approximate dates at which they flourished:—PYTHAGORAS, 530 B. C.; EUCLID, 290 B. C.; ARCHIMEDES, 250 B. C.; APOLLONIUS, 230 B. C.; HIPPARCHUS, 150 B. C.; HERON (HERO), 120 B. C.; DIOPHANTUS, 275 A. D.; THEON, 380 A. D.; PROCLUS, 450 A. D.

INTRODUCTION.

Before entering upon a discussion of the methods employed by the ancient Greeks for solving Quadratic Equations, a brief summary should be made of the mathematical knowledge which they possessed in historic times. Various forms of reckoning, including finger-reckoning, pebble-reckoning, and some use of the sand-board ($'a\beta a'\xi$) especially characteristic of the Attic Greeks, had by the time of Pythagoras given place to a well defined system of notation and computation by means of symbols. These consisted of the letters of the regular Greek alphabet, with three additions from an older alphabet, which were used to make up a decimal system of notation.* Although operations with these symbols were cumbrous and complicated, they were possible and the ordinary operations of addition, subtraction, multiplication, and division were quite fully developed.† They had made some progress in the theory of numbers which was denoted by the term $'\alpha\rho\iota\theta\mu\eta\tau\iota\kappa\eta'$ as contrasted with $\lambda o\gamma\iota\sigma\tau\iota\kappa\eta'$ the art of calculation. Their knowledge of Algebra, as an abstract science, was almost nothing, until after the time of Christ, although they had a slight conception of it, perhaps, from the Egyptians who were familiar with simple equations long before the time of Pythagoras.‡ But the Greeks had made unusual progress in Geometry, and of the problems of Algebra which can easily be given a geometric interpretation and solved by methods essentially geometrical, they had a very considerable knowledge which we shall investigate a little more in detail. Accordingly we shall take up the Greek methods of solving quadratic equations under two distinct heads, geometrical or constructive methods, and methods purely algebraic.

GEOMETRIC METHODS.

Pure Quadratics. 1. *Square Root.* Although not formulated by the Greeks in this way we may properly look upon their work in finding square roots of numbers as the solution of the pure quadratic $x^2=a$, and accordingly as the first work which they did in the solution of quadratic equations.

The Pythagoreans are credited with the first knowledge of the fact that in a square

$$\text{diagonal} : \text{side} :: \sqrt{2} : 1,$$

and of the method by which it was established, but it was kept a profound secret from their contemporaries.§ Although there was this geometrical

* See Cantor, pp, 110-119; Cajori, p. 64; Heath, *Archimedes*, p. lxix; Nesselmann, p. 78, et seq.

† Numerous problems given in Gow; Heath, *Archimedes*; Tannery; Hankel.

‡ See Cantor, pp. 37, 38, for numerous examples of equations from the Ahmes papyrus.

§ It is interesting to note the story that the Pythagorean who first divulged this knowledge of the irrationals perished in a shipwreck as an evidence of the displeasure of the gods. A similar story is told of the discloser of the knowledge of the dodecahedron. See Allman, pp. 25, 47.

equivalent for $\sqrt{2}$, they also recognized its irrationality but had no numerical expression for it, because they did not recognize irrationals as members of their number system; when they occurred they were rejected in the same way that imaginaries were in later centuries.* Lines were the indispensable symbols for irrationals because they avoided the necessity for numerical expression. But in the further development of Geometry, especially about the time of Euclid, it became necessary to obtain approximate values for some square roots, and the results and methods by which they were obtained are worth consideration.

Archimedes, perhaps the greatest mathematician of antiquity, is the first to give us any number of them. But unfortunately he gives only results and no suggestion of the method by which he reached them, nor is there any example of the actual complete calculation of a root extant by anyone before the time of Christ. Archimedes in his *Circuli Dimensio*† in dealing with the problem of finding an approximate value for the ratio of circumference to diameter says that $\sqrt{3}$ lies between $\frac{1351}{780}$ and $\frac{265}{153}$, that is, between 1.7320513 and 1.7320327. Now $\sqrt{3}$ correct to seven decimal places is 1.7320508, so that his approximation is remarkably accurate, his upper limit being exactly equal to six places. Many ingenious theories have been proposed to explain how Archimedes could secure this result, among them a method by continued fractions, one by an approximation in the form of a series of fractions, Theon's method of sexagesimal fractions based on Euclid (illustrated later, see p. 7), and others.‡ But we have no positive knowledge as to how he secured this result. If it were for $\sqrt{3}$ alone we might accept the theory that his only method consisted in guessing at the result and gradually making successive guesses more nearly accurate. But this seems highly improbable in view of the following results which occur in the same connection in Archimedes.§ (I have added the correct results to two places in the third column.)

3013 3/4	is greater than	$\sqrt{9082321}$	3013.69
1838 9/11	is greater than	$\sqrt{3380929}$	1838.73
1009 1/6	is greater than	$\sqrt{1018405}$	1009.16
2017 1/4	is greater than	$\sqrt{4069284\ 1/36}$	2017.247
591 1/8	is less than	$\sqrt{349450}$	591.14
1172 1/8	is less than	$\sqrt{1373943\ 33/64}$	1172.15
2339 1/4	is less than	$\sqrt{5472132\ 1/16}$	2339.26

* See Cantor, p. 170.

† Prop. 3. Heiberg, *Archimedes*, p. 262 et seq. Heath, p. 93.

‡ Heath refers to Gunther, *Die quadratischen Irrationalitäten der Alten und deren Entwickelungsmethoden*, Leipzig, 1882, as an exhaustive paper discussing in detail all the hypotheses offerred up to 1882 for finding $\sqrt{3}$, including those of Zeuthen, Tannery, DeLagny, Heilermann, and Rodet.

§ Prop. 3. Heiberg, *Archimedes*, p. 262 et seq. These values occur in finding the value of π by the successive steps necessary in inscribing in a circle a regular polygon of 96 sides.

These are very close approximations and those are especially interesting where the number whose square root is sought is itself fractional. It is quite clear from these examples that Archimedes must have had a definite method to secure such accurate results with these large and varied numbers.* It is worth noting in passing that the value of π which he secured by means of these numbers was between 3 1/7 and 3 10/71, that is, between 3.1429 and 3.1408.†

A century and a quarter later Heron used as a formula for computing square root, $\sqrt{a^2 \pm b}$ is approximately $a \pm \frac{b}{2a}$.‡ He gives as roots determined thus:

$\sqrt{50}$ is $7+1/14$, *i. e.* 7.071.
$\sqrt{63}$ is $8-1/16$, *i. e.* 7.937.
$\sqrt{75}$ is $8+11/16$, *i. e.* 8.687.

The first two are correct to three places, but the third where the "*b*" is not unity is correct only to the first place, the true value being 8.660. Heron's method is probably based on Euclid, being an adaptation of the process given in the next paragraph. He also used for greater accuracy $a \pm \frac{b}{2a} > \sqrt{a^2 \pm b} > a \pm \frac{b}{2a \pm 1}$, by which $\sqrt{75}$ would be between 8.687 and 8.647. From the fact that Archimedes' results all appear as "greater than" or "less than" the real value it would seem that he might have used some similar formula; but of course Heron's is not accurate enough for the results given. Heron also gives 7/5 for $\sqrt{2}$ (correct value 1.414+), and 26/15 for $\sqrt{3}$, *i. e.* 1.733 (correct value 1.732+). In his *Stereometrica* is the first known attempt to express the square root of a negative number. Without method or comment $\sqrt{81-144}$ is stated to be 8 less than 1/16!

The last method of finding square root we shall give is found in the works of Theon,§ very late, in the fourth century after Christ. It makes use of the sexagesimal system of angles of the Babylonians but is not essentially different from our present method. Although based on Euclid, Theon's language would indicate that the method itself is comparatively new. As showing the antiquity of our method in its essence and illustrating the pro-

* Heath says "There is no doubt that in obtaining the integral portion of the square root" he used the method of Theon, illustrated in the next paragraph. Gow: "It is clear that Archimedes did not use Theon's method!" And there are various shades of opinion between these extremes. See previous references to Gunther. Also Cantor, p. 301 et seq. Heath, *Archimedes*, p. lxxx, gives in detail Hultsch's theory of "tentative assumptions" ingeniously worked out for many of these results, with several circumstances leading to its probability. See also Nesselmann, p. 108 et seq.

† Heiberg, *Archimedes*, Vol. I, p. 270.

‡ Fink, p. 70.

§ In his commentary on Ptolemy's *Almagest* (A. D. 125).

cedure of at least the later Greeks we give the following example paraphrased from Theon.*

"I ought to mention how we extract the approximate root of a quadratic which has only one irrational root. We learn the process from Euclid, II, 4, where it is stated 'if a straight line be divided at any point, the square of the whole line is equal to the square of both the segments together with twice the rectangle contained by the segments.' So with a number like 144, which has a rational root, as the line $\alpha\beta$ (see Fig. I) we take a lesser square say 100, of which the root is 10, as $\alpha\gamma$. We multiply 10 by 2 because there are two rectangles, and divide 44 by 20. The remainder 4 is the square of $\beta\gamma$ which must be 2. Let us try the number 4500 (see Fig. II), of which the root is 67° 4′ 55″. Take a square $\alpha\beta\gamma\delta$ containing 4500 degrees ($\mu\acute{o}\iota\rho\alpha\iota$). The nearest square number is 4489, of which the side (root) is 67°. Take $\alpha\eta$=67° and $\alpha\epsilon\xi\eta$ the square of $\alpha\eta$. The remaining gnomen $\beta\xi\delta$ contains 11° or 660′. Now divide 660′ by 2 $\alpha\eta$, *i. e.* by 134. The quotient is 4′. Take $\epsilon\theta$, $\eta\kappa$=4′ and complete the rectangles $\theta\xi$, $\xi\kappa$. Both these rectangles contain 536′ (268′ each). There remains 124′=7440″. From this we must subtract the square $\xi\lambda$ containing 16″. The remaining gnomen $\beta\lambda\delta$ contains 7424″. Divide this by 2 $\alpha\kappa$=134° 8′. The quotient is 55″. The remainder is 46″ 40‴, which is the square of $\lambda\gamma$, of which the side is 55″ nearly enough".

Fig. I.

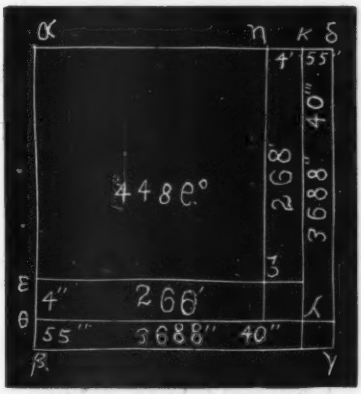

Fig. II.

Theon also gives another example with a figure for $\sqrt{2^\circ\ 28'}$ which he finds to be 1° 34′ 15″. This is correct to .8′. The procedure is the same as above. Theon concludes: "When we seek a square root we first take the root of the nearest square number. We then double this and divide with it the remainder reduced to minutes, and subtract the square of the quotient. Then we reduce the remainder to seconds, and divide by twice the degrees and

* Gow, p. 55. See also Cantor, pp. 460-461; Heath, *Archimedes*, p. lxxvi.

minutes (of the whole quotient). We thus obtain nearly the root of the quadratic." Except for the sexagesimal notation this does not sound very different from the rule in our elementary texts today.

By this process Ptolemy gives $\sqrt{3}=\frac{103}{60}+\frac{55}{60^2}+\frac{23}{60^3}$, which in our notation is 1.7320509 and so correct to 6 places.

2. *Constructive methods.* If the pure quadratic is considered in the form "$x^2=ab$" it means in geometrical language to find a square equivalent to a given rectangle, and this was solved by Euclid and perhaps his predecessors, in the two propositions, "to construct a square equal to a given rectilinear figure"* and "to two given straight lines to find a mean proportional"† being the same methods given in our elementary geometries today.

If the quadratic is in the form "$x^2=pa^2$" it becomes geometrically the problem of the multiplication of the square, which is solvable by the Pythagorean theorem.‡ In its special form "$x^2=2a^2$" it becomes the problem of the duplication of the square, which we have already noted was first known by the Pythagoreans. Perhaps its successful solution suggested the more famous problem of antiquity, the duplication of the cube.

Affected Quadratics. By geometrical methods the Greeks were able to solve any equation of the type $x^2+px+q=0$ where a real solution was possible, although they did not consider it thus generally. Rather they considered quadratics under three forms,

$$\text{(a)}\quad x^2+px=q$$
$$\text{(b)}\quad x^2+q=px$$
$$\text{(c)}\quad x^2=px+q$$

due to the fact that the treatment of them was geometric. Their solution consisted principally in applying theorems on areas, one of the most powerful methods on which Greek geometry relied, or on proportion, in which they were also well versed.

The procedure may be shown by a typical example, of the first type,

$$x^2+ax=b.\S$$

Expressed geometrically, this would be "To the segment $AB=a$ (see Fig. III) apply the rectangle DH, of known area, b, in such a way that CH shall be a square." The figure shows that for $CK=a/2$

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* Euclid, II, 14. Heiberg, *Euclid*, Vol. I, p. 160.
† Euclid, VI, 13. Heiberg, *Euclid*, Vol. II, p. 110.
‡ Euclid, I, 47. Heiberg, *Euclid*, Vol. I, p. 110.
§ Fink, p. 79. Zeuthen, p. 36 et seq., gives several more.

$FH=x^2+2(a/2)x+(a/2)^2=b^2+(a/2)^2$.

But by the Pythagorean theorem

$$b^2+(a/2)^2=c^2,$$

whence $EH=c=a/2+x$

and $x=c-a/2=BC$.

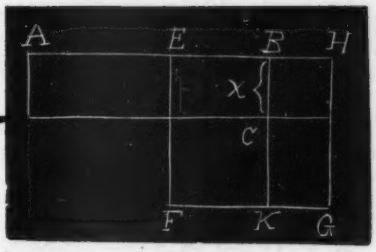


Fig. III.

In the same way Euclid solves all problems of this form and says that b must be greater than $a/2$ in $\sqrt{b^2-(a/2)^2}$ in order to give a solution. Thus imaginary roots are excluded.

The equation just solved is simply another way of expressing $x(x+p)=q$, which stated in Euclidean language would be "To produce a given line p to length $p+x$ so that the rectangle between the whole line so produced and the part produced, *i. e.* $x(x+p)$ shall be equal to a given figure q." Similarly, $x^2=a(a-x)$ of the third type is Euclid's proposition "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment."* And Euclid finds $x=\sqrt{a^2+(a/2)^2}-a/2$.† It is certain that this particular problem, and probably others similar, were solved much earlier than Euclid, even by the Pythagoreans.‡

Their general method of solution by areas may be stated as follows. The problem was to apply to a given line a rectangle or more generally a parallelogram so that it would either contain a given area, or be greater or less than the given area by a constant. For these three conditions there arose probably even among the Pythagoreans the names παραβολή' (parallel to, application, equal), 'υπερβολή' (excess), and 'ελλειψις (falling short).§ After the time of Archimedes, however, Apollonius took these names for the conic sections because they were appropriate to the distinctive properties of the parabola, hyperbola, and ellipse as he defined them.

The solution of affected quadratics by proportionality of lines was more general, but a later development probably not extensively used much before Euclid, while we have seen that the one by areas antedated him two centuries or more. In his sixth book Euclid gives by proportion the equivalent of the solution of the general equation $ax^2 \pm \frac{b}{c}x=S$ subject to the condition for real roots.‖

* Euclid II, 11. Heiberg, *Euclid*, Vol. 1, p. 152.

† Cantor, pp. 249-250.

‡ Heath, *Euclid*, Vol. I, note to II, 5, p. 384, and II, 11, p. 403.

§ For antiquity of these terms see Proclus' note to Euclid I, 44, in Allman, p. 24, and Heath, *Euclid*, Vol. I, p. 343, "These things (*i. e.* method of areas) are ancient and are the Discoveries of the Muse of the Pythagoreans."

‖ VI, 27, 28, 29, 30. Heiberg, *Euclid*, Vol. II, pp. 158-172. Discussed in their algebraic bearing in Heath, *Euclid*, Vol. 2, p. 260 et seq. Cantor, p. 252.

Apollonius did the same thing by means of the conics, especially with the aid of the equation $y^2=px\pm\frac{p}{a}x^2$, the equation which he uses to express the fundamental property of a central conic.*

We conclude accordingly that the Greeks before the beginning of the Christian era were able to solve any equation of the second degree, having two essentially different coefficients, *geometrically* for real positive roots.

Algebraic Methods.

Beginnings. A great step forward in mathematical reasoning was made when the Greeks were enabled to divorce their Algebra from their Geometry and reason abstractly, without necessary connection with the concrete concepts of geometry. Only then did they commence to develop a real science of algebra as a distinct subject. As a separate science among the Greeks it seems to have had its beginnings at least as early as the second century before the Christian era, but it did not reach a comparatively consistent development until two or three centuries later under Diophantus, by far the most important name in the consideration of Greek Algebra.

Arabian authorities state that Hipparchus wrote a treaties on the solution of quadratic equations, but no traces of it exist today, and we have no way of knowing whether it was any advance over the geometrical methods we have been considering.†

Heron a little later was the first to adopt the algebraic method, demonstrated the first ten or twelve propositions of Euclid, Book II, by means of lines only, without reference to areas, and dealt with them analytically as representations of pure numbers. He seems to have solved the affected quadratic equation $ax^2+bx=c$ by completing the square, but the evidence is not conclusive.‡

Diophantus. There is much dispute as to the date of Diophantus but probably it is safe to say he flourished in the last half of the third century of our era.§

La Grange speaks of him as "l'inventeur de l'Algebre"‖ while Tannery is unwilling to credit him with being anything more than a compiler. Nesselmann takes the intermediate view that the greater part of his propositions and ingenious methods are his own.¶ It seems fair to say that he did for Algebra what Euclid did for Geometry, organizing previous knowledge and adding much due to his own genius. His principal treatise is on "Arithmetic" (*Ἀριθμητικῶν βιβλί'α ἰλ*) in six books of which a portion is

* Heath, *Apollonius*, p. cx.

† Cantor, p. 346.

‡ See Cantor, p. 376 et seq.

§ Fully Discussed in Heath, *Diophantus*, pp. 3-17.

‖ La Grange, Vol. 7, p. 219.

¶ Nesselmann, p. 477. For full discussion of question of Diophantus' originality see Heath, *Diophantus*, Chap. VII, Cantor, p. 427 et seq.

lost; it is the first treatise on Algebra extant.* He was the first to use symbols for operations and for an unknown quantity, although he does not claim this symbolism is original. The unknown quantity was called ὁ ἀριθμός and its symbol was ς' or $\varsigma^{o}{}'$, plural $\varsigma\varsigma$ or $\varsigma\varsigma^{oι}$. The square of the unknown was δύναμις and its symbol $\widehat{\delta^{u}}$. Symbols were used for as high as the sixth power of the unknown.† One of his equations looks like this

$\widehat{\kappa^{u}}$	β	$\widehat{\delta^{u}}$	$\bar{\alpha}$	$\iota'\varsigma\eta$	$\varsigma\varsigma\widehat{οις}$	$\overset{o}{\iota}$	ϕ	$\mu^{o}{}'$ $\bar{\iota\beta}$
x^3	2	x^2	1 equals	x		4 minus	units	12

i. e. $2x^3+x^2=4x-12$. The part of Diophantus' work which deals with the solution of determinate quadratics is lost and we have little information as to its contents. Most of his extant work deals with the solution of indeterminate equations, including quadratics. His work shows deficiency in generalization but unusual ingenuity in the manipulation of processes, especially necessary since he never uses more than one symbol for the unknown, and other unknowns must be expressed in terms of it.

A single solution is sufficient for any set of conditions. Even when they lead to a quadratic he gives but one root, showing his ignorance of their true nature.‡ He refuses negative or imaginary roots as "impossible" or "absurd," but distinguishes between rational and irrational roots seeking only the former. His fundamental basis of classification of equations is not according to degree but *number of terms* when reduced to simplest form, *i. e.*, $ax^n=c$ is a "simple" equation whatever the power of x, while $ax^2+bx=c$ is a "mixed" equation with two terms. In what follows we take that part of his work, for the most part, which deals with what we know as quadratic equations, although, as pointed out, this distinction is not always made by him.

Pure Quadratics. In Definition 11 of the First Book§ he gives his method for solving problems leading to pure quadratics. "If a problem leads to an equation containing the same powers of the unknown (ἐπὶ τὰ αὐτά) on both sides but not with the same coefficients (μὴ ὁμοπλήθη) you must deduct like from like until only equal terms remain. But when on one side or both some terms are negative, (ἐν ἑλλείψεσι) you must add the negative terms to both sides till all the terms are positive (ἐνυπάρχοντα) and then deduct as before stated. If he comes to an equation such as $x^2=ax$ he merely divides by the factor $x=0$ which he does not regard as a solution, *e. g.*,

* Diophantus announces 13 books. The seven existing mss. are in six books except one, which is in seven but has the same material. Probably we have the bulk of the original 13 books, with some sections lost (notably those on solution of determinate quadratics) and the remainder rearranged by later editors. See Nesselmann, p. 265 et seq.; Gow, p. 101-102; Heath, *Diophantus*, Chap. II.

† For symbolism see Nesselman, pp. 294-296; Heath, *Diophantus*, Chap. IV; Cantor, p. 439.

‡ Hankel (p. 162) thinks this is a relic of the old geometric practice. Gow suggests that it was because Algebra was the invention of practical men who needed only the one solution.

§ Tannery, Vol. I, p. 14.

"$20x=10x^2$ whence $x=2$."*

Affected Quadratics. In Definition 11, Diophantus promises to give his method for solving "mixed" or affected quadratics. He continues: "We must contrive always, if possible, to reduce our equations so that they may contain one single term equated to one other. But afterward we will explain to you also how, when two terms are left equal to a third, such a question is solved." That is, "reduce to the form $x^2=a$ if possible. Later we will give a method for the complete quadratic $x^2+ax=b$." But this method, if ever written, is in the part which is lost. Without the method of solution he states numerous results, *e. g.*, "$84x^2+7x=7$, whence x is found equal to $\frac{1}{4}$"†; "$630x^2+73x=6$, whence the root is rational,"‡ and many similar ones. When a root is irrational he sometimes gives an approximation to it.§ It is unfortunate that he nowhere gives an example of the complete solution of a single equation, but the form of his solution of equations of this type

$$ax^2+bx+c=0$$

is usually

$$x=\frac{(1/2)b\pm\sqrt{(1/4)b^2-ac}}{a}.$$

This is exactly the form it would have, had he completed the square after first multiplying the equation through by the coefficient of the x^2 term. The consensus of opinion seems to be that this is the way in which he arrived at his results.‖ From the variety of equations which arise in Diophantus' work we must believe that he had a general method of solution for all *determinate* quadratics, sufficient to determine one real root if such existed.

No sufficient idea can be given of the general methods he used in resolving *indeterminate* equations into solvable forms¶, but two examples will be given to show his ingenuity and some of his methods of attack.

A common method of attack is that of *tentative assumptions*, in which a value is shrewdly assumed and then altered as occasion arises until it fits the conditions of the problem. It frequently involves the use of his single symbol for the unknown quantity in different senses. For example, in book four° the problem is "to find three numbers such that their sum is a square

* I. 31; Tannery, Vol. I, p. 68, and many others in first book.

† VI, 6. Tannery, p. 404.

‡ VI, 8. Tannery, p. 408.

§ *E. g.* V, 30. Tannery, p. 384.

‖ See Nesselmann, p. 319; Cantor, p. 443 et seq; Heath, *Diophantus*, pp. 90-92.

¶ Hankel (pp. 164-165) says it is difficult after solving 100 of Diophantus' problems to have any idea how to attack the 101st. Nesselmann gives an unsatisfactory classification and discussion of methods of solution (pp. 315-328) and says himself that a complete discussion would mean copying Diophantus' entire work. Heath disagrees with both and devotes Chap. V of his book on Diophantus to a fairly satisfactory classification and discussion.

° IV, 17. Tannery, pp. 222-225. Gow, p. 117.

and the square of any number plus the following one is a square." He assumes the three numbers first as $(x-1)$ $(4x)$ $(8x+1)$ where $(x-1)^2+4x$ and $(4x)^2+(8x+1)$ are both square numbers. Two of the conditions are thus satisfied, but the third is that the sum of the numbers, $13x$, must also be a square number. He says: "Take $13x=x^2$ with some square coefficient e. g. $13x=169x^2$. Then $x=13x^2$." A new use of "x" is thus introduced and $13x^2$ is substituted for the original x, the numbers now being, $13x^2+1$, $52x^2$ and $104x^2-1$. A fourth condition remains, viz, that

$$(104x^2+1)^2+(13x^2-1)$$

shall be a square. Diophantus then takes this expression equal to

$$x^2(104x^2+1)^2$$

and finds $x=\frac{111}{104}$, and substitutes this value, finally getting his three numbers

$$\frac{170989}{10816},\ \frac{640692}{10816},\ \frac{1270568}{10816}\ldots$$

Of course this has all been expressed in modern notation. His last number, for example, is expressed

$$\frac{\alpha}{\rho\kappa\xi} : \frac{\omega\iota\varsigma}{\phi\xi\eta}.$$

Another frequent artifice which he uses is a method of *limits*. If he wishes to find a square number, between 10 and 11 for example, he multiplies these by successive squares until a square number lies between the product; thus between 40 and 44 or 90 and 99 no (rational) square lies, but between 160 and 176 is 169; hence $x^2=\frac{169}{16}$ will lie between the proposed limits. This is made use of in the problem: "To divide 1 into two parts such that if 3 be added to the first part and 5 to the other, the product of the two sums shall be a square."* If one part be $x-3$ the other is $4-x$. Then $x(9-x)$ must be a square. Suppose it equals $4x^2$, then $x=\frac{9}{5}$. But this will not suit the original assumption since x must be greater than 3 and less than 4. Now 5 is $4+1$. Hence what is wanted is to find a number y^2+1, such that $\frac{9}{y^2+1}$ is greater than 3 and less than 4. For such a purpose y^2 must be less than 2 and greater than $1\frac{1}{4}$. "I resolve these expressions into square

* IV, 31; Tannery, pp. 278-283; Gow, p. 119. For this method see also Nesselmann, p. 318; Heath, *Diophantus*, pp. 115-120.

fractions" says Diophantus, selecting $\frac{1}{6}\frac{2}{4}\frac{8}{4}$ and $\frac{8}{6}\frac{0}{4}$ between which lies the square $\frac{1}{6}\frac{0}{4}\frac{0}{4}$ or $\frac{2}{1}\frac{5}{6}$. He then puts $x(9-x)=\frac{25x^2}{16}$ instead of $4x^2$, whence $x=\frac{1}{4}\frac{4}{1}\frac{4}{1}$, and the two numbers are $\frac{9}{4}\frac{1}{1}$ and $\frac{2}{4}\frac{0}{1}$.

In general we may say that the type of indeterminate quadratic equations which he considers fully are limited to the case where one or two functions of the unknown in the form Ax^2+Bx+C must be a rational square, and are only fully treated in cases where the "C" vanishes. Otherwise his solution depends upon the particular equation involved, and consists of ingenious transformations or assumptions of the unknown.† The most characteristic feature of his work is the extraordinary artifices which he employs.

Such then are the principal methods known to the ancient Greeks for solving quadratic equations. In the classic period they had a comparatively complete knowledge of methods of solution from the purely geometric standpoint, practically none algebraically. Even in later times their knowledge of the algebraic solution was faulty and cramped, showing no true conception of the real nature of the quadratic, only familiarity with ingenious methods of manipulating particular equations. Considering the exceptionally high attainments of the Greeks in geometry as well as other branches of learning and culture, it seems a little strange that they did not make greater advances in algebra. This was reserved for the Hindoos some centuries later. Nevertheless it is instructive to trace out some of their attainments in this particular line in order to note their limitations as well as their achievements.

† Heath, *Diophantus*, p. 115.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

344. Proposed by V. M. SPUNAR, Cleveland, Ohio.

Given $x^7-5x^2y^4=-1506\ldots(1)$, and $y^5-3xy=103\ldots(2)$; find the values of x and y.

Solution by E. B. ESCOTT, Ann Arbor, Mich.

Drawing the graphs of the two equations we see that the only real intersection is for x between -2 and -2.84, and y between 2.4 and 2.53.

A few trials give the values to a few decimals

$$x=-2.4168, \quad y=2.43359.$$

Also solved by the Proposer.

345. Proposed by E. B. ESCOTT, Professor of Mathematics, University of Michigan.

Solve the simultaneous equations:

$$x^3+y^2z+zw^2-xz^2-2wxy=a\ldots(1).$$
$$y^3+z^2w+wx^2-yw^2-2xyz=b\ldots(2).$$
$$z^3+w^2x+xy^2-zx^2-2yzw=c\ldots(3).$$
$$w^3+x^2y+yz^2-wy^2-2zwx=d\ldots(4).$$

Solution by the PROPOSER.

If we put the determinant

$$\begin{vmatrix} x & y & z & w \\ w & x & y & z \\ z & w & x & y \\ y & z & w & x \end{vmatrix} = D\ldots(5),$$

we see that the minors of the first row are the first members of the given equations. Therefore, the minors of x, y, z, w in the determinant, D, are equal to a, b, c, d. Therefore the determinant

$$\begin{vmatrix} a & -b & c & -d \\ -d & a & -b & c \\ c & -d & a & -b \\ -b & c & -d & a \end{vmatrix} \ldots(6),$$

is the reciprocal determinant of D and is therefore equal to D^3. Also by the

properties of reciprocal determinants the minors of the first row in (6) with their proper signs are equal to corresponding terms of the first row in (5) multiplied by D^2, *i. e.*

$$\left.\begin{array}{l} \begin{vmatrix} a & -b & c \\ -d & a & -b \\ c & -d & a \end{vmatrix} = xD^2 \\ -\begin{vmatrix} -d & -b & c \\ c & a & -b \\ -b & -d & a \end{vmatrix} = yD^2 \end{array}\right\} \ldots (7),$$

etc. Solving (7) we have. after making a few changes of sign,

$$x = \frac{\begin{vmatrix} a & b & c \\ d & a & b \\ c & d & a \end{vmatrix}}{\begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{vmatrix}} = \frac{3}{8} \triangle.$$

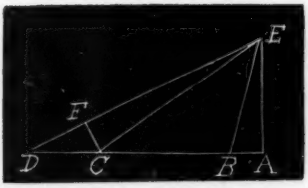
$$y = \frac{\begin{vmatrix} d & b & c \\ c & a & b \\ b & d & a \end{vmatrix}}{\triangle} \qquad z = \frac{\begin{vmatrix} d & a & c \\ c & d & b \\ b & c & a \end{vmatrix}}{\triangle} \qquad w = \frac{\begin{vmatrix} d & a & b \\ c & d & a \\ b & c & d \end{vmatrix}}{\triangle}$$

GEOMETRY.

372. Proposed by DANIEL KRETH, Oxford, Iowa.

In the right triangle ADE right angle A, are given: $AB=9$, $BC=280$, $CD=35$, angle AEB=angle CED; required the distance AE.

I. Solution by A. H. HOLMES, Brunswick, Me.



For $AB=9$, $BC=280$, and $CD=35$, the points B and C must be on the line AD. Otherwise the problem would be indeterminate. Let $AE=x$. Then $DE=\sqrt{(x^2+324^2)}$, $CE=\sqrt{(x^2+289^2)}$, and $BE=\sqrt{(x^2+9^2)}$.

From C let fall perpendicular CF to DE at F. To find EF we have

$$\sqrt{(x^2+324^2)} : \sqrt{(x^2+289^2)}+35 = \sqrt{(x^2+289^2)}-35 : \frac{x^2+289^2-35^2}{\sqrt{(x^2+324^2)}}.$$

$$\therefore EF=\frac{2x^2+324^2+389^2-35^2}{2\sqrt{(x^2+324^2)}}.$$

Then since $CED=AEB$ we have:

$$\sqrt{(x^2+289^2)} : \frac{2x^2+324^2+289^2-35^2}{2\sqrt{(x^2+324^2)}}=\sqrt{(x^2+9^2)} : x.$$

Squaring and reducing, $x^4-13259.64x^2=620781241.4125$.
$\therefore x=180.03+$

II. Solution by B. KRAMER, E. M., and J. E. SANDERS, Weather Bureau, Columbus, Ohio.

$\frac{\text{Area } AEB}{\text{Area } CED}=\frac{9}{35}$. Also the triangles having one angle equal,

$$\frac{x\sqrt{(x^2+9^2)}}{\sqrt{[(x^2+17^4)(x^2+18^4)]}}=\frac{9}{35}.$$

Squaring, $35^2x^2(x^2+9^2)=9^2(x^2+17^4)(x^2+18^4)$, or
$2^3.11.13x^4-17^2.2^2.9^4x^2-17^4.9^4.2^4.3^4=0$.

Solving for x^2, $x^2=\frac{17^2.9^4\pm\sqrt{(17^4.9^8+17^4.9^4.2.3^4 2^4.11.13)}}{2.11.13}$

$$=\frac{17^2.9^4\pm17^2 9^3\sqrt{(35^2)}}{2.11.13}=\frac{17^2 9^3(9\pm35)}{2.11.13}.$$

Taking the + sign, the one real solution for x^2 is

$$x^2=\frac{17^2.9^2.3^2.2^2.11}{2.11.13}=\frac{17^2.9^2.3^2.2}{13}. \quad x=\pm17.9.3\sqrt{\tfrac{2}{13}}=\pm180.0346. \quad \text{KRAMER.}$$

An easier and more general result is obtained by using the tangent formula, thus:

$$\tan AEB=9 : x \ldots (1).$$

$$\tan DEC=\frac{\tan DEA-\tan CEA}{1+\tan DEA\times\tan CEA}=\frac{\frac{324-289}{x}}{1+\frac{324.289}{x^2}} \ldots (2).$$

Angles (1) and (2) being equal:

$$\frac{35x^2}{x(x^2+17^2.18^2)}=\frac{9}{x}, \text{ or } 26x^2=3^2.17^2.18^2;\ x^2=\frac{3^2.17^2.18^2}{26}.$$

$$x=\pm\frac{3.17.18}{\sqrt{26}}=\pm 3.17.9\sqrt{\tfrac{2}{13}}=\pm 180.0346.$$

SANDERS AND KRAMER.

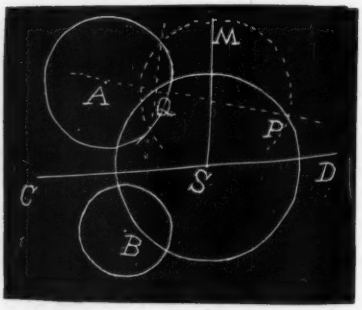
Also solved by V. M. Spunar and J. Scheffer.

373. Proposed by S. LEFSEHETZ, East Pittsburg, Pa.

Draw a circle passing through a given point and orthogonal to two given circles.

Solution by A. R. MAXSON, Columbia University, New York City.

Let P be the given point, and A and B the centers of the given circles. Draw CD, the radical axis of circles A and B. The required circle must have its center in CD. Again, remembering that the common chord of circle A and a circle described on AP as diameter cuts AP in a point Q, inverse of P with respect to circle A, construct Q and draw MS, the perpendicular bisector of QP cutting CD in S. S is the center of a circle of radius SP that passes through P and is orthogonal to circles A and B. We know that any circle through two points inverse with respect to a given circle is orthogonal to that circle. MS is, in our case, the locus of centers of circles through the inverse points P and Q. In particular, then, the circle of center S, S being in CD, must satisfy all the conditions of the problem.



Also solved by J. Scheffer, V. M. Spunar, S. G. Barton, C. N. Schmall, and the Proposer.

CALCULUS.

296. Proposed by C. N. SCHMALL, New York City.

Two currents C_1 and C_2 produce deflections ϕ_1, ϕ_2, respectively, in a tangent galvanometer. When is $(\phi_1-\phi_2)$ a maximum?

I. Solution by the late G. B. M. ZERR, Ph. D.

Let $G=2\pi n/r$=principal constant of the galvanometer. H=earth's horizontal magnetic force. $H/G=1/C$. Then $\tan\phi_1=CC_1$, $\tan\phi_2=CC_2$.

$\therefore u=\phi_1-\phi_2=\tan^{-1}(CC_1)-\tan^{-1}(CC_2)$.

$$\frac{du}{dC_1}=\frac{C}{1+C^2C_1^2},\quad \frac{du}{dC_2}=-\frac{C}{1+C^2C_2^2},$$

$$\frac{d^2u}{dC_1^2}=-\frac{2C^3C_1}{(1+C^2C_1^2)^2},\quad \frac{d^2u}{dC_2^2}=\frac{2C^3C_2}{(1+C^2C_2^2)^2},\quad \frac{d^2u}{dC_1dC_2}=0.$$

Now $\frac{d^2u}{dC_1^2}\frac{d^2u}{dC_2^2}<\frac{d^2u}{dC_1dC_2}$.

Hence there is no maximum unless $C_1=\infty$, $C_2=-\infty$.

II. Solution by V. M. SPUNAR, Chicago, Ill., and the PROPOSER.

By the principle of the instrument, we have

$$\frac{\tan\phi_1}{\tan\phi_2}=\frac{c_1}{c_2}.\quad \therefore\frac{\tan\phi_1-\tan\phi_2}{\tan\phi_1+\tan\phi_2}=\frac{c_1-c_2}{c_1+c_2}\ldots(1).$$

But by trigonometry,

$$\tan\phi_1-\tan\phi_2=\frac{\sin(\phi_1-\phi_2)}{\cos\phi_1\cos\phi_2},\quad \tan\phi_1+\tan\phi_2=\frac{\sin(\phi_1+\phi_2)}{\cos\phi_1\cos\phi_2}.$$

Hence (1) becomes

$$\frac{\sin(\phi_1-\phi_2)}{\sin(\phi_1+\phi_2)}=\frac{c_1-c_2}{c_1+c_2},\text{ or }\sin(\phi_1-\phi_2)=\left(\frac{c_1-c_2}{c_1+c_2}\right)\sin(\phi_1+\phi_2).$$

Hence, $(\phi_1-\phi_2)$ is a maximum when $\sin(\phi_1+\phi_2)$ is maximum; that is, when $\sin(\phi_1+\phi_2)=1$; that is, $\phi_1+\phi_2=\frac{1}{2}\pi$.

297. Proposed by PROF. L. C. WALKER, Socorro, New Mexico.

A square hole $2s$ on a side is cut through an ellipsoid, axes $2a$, $2b$, $2c$, the axis of the hole coinciding with the axis $2c$ of the ellipsoid. Find (1) the volume, and (2) the surface removed.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

The volume $V=8c\int_0^s dx\int_0^s dy\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}$

$$=\frac{4c}{b}\left[\frac{bs}{a}\int_0^s\sqrt{\frac{a^2(b^2-s^2)}{b^2}-x^2}+\frac{b^2}{a^2}\int_0^s(a^2-x^2)\sin^{-1}\frac{as}{b\sqrt{(a^2-x^2)}}\partial x\right.$$

$$=\frac{4bcs}{3a^2}(3a^2-s^2)\sin^{-1}\frac{as}{b\sqrt{(a^2-s^2)}}+\frac{4acs(3b^2-s^2)}{3b^2}\sin^{-1}\frac{bs}{a\sqrt{(b^2-s^2)}}$$

$$+\tfrac{8}{3}abc\sin^{-1}\frac{s^2}{(a^2-s^2)^{\frac{1}{2}}(b^2-s^2)^{\frac{1}{2}}}+\frac{10cs^2}{3ab}\sqrt{[a^2b^2-(a^2+b^2)s^2]}.$$

The surface $S=\dfrac{4}{ab}\displaystyle\int_0^s dx\int_0^s dy\sqrt{\frac{a^4b^4-b^4(a^2-c^2)x^2-a^4(b^2-c^2)y^2}{a^2b^2-b^2x^2-a^2y^2}}$.

The integration of this leads to elliptical functions.

NOTE. If any one will complete the solution of the second part of this problem we shall be pleased to publish it. ED. F.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

NOTE. Dr. Whalen, of the University of Illinois, has consented to edit this department in the future. Will contributors please send their problems and solutions to him?

174. Proposed by B. KRAMER, Student, University of Pittsburg, Pittsburg, Pa.

Find a general solution of $x(x+a)=y^2$, a, x, and y being integers. Given a, required to find x to satisfy conditions.

I. Solution by PROFESSOR F. L. GRIFFIN, Ph. D., Williams College.

Let d be any integer contained in a an odd number of times. Then the general solution is: $x=(a-d)^2/4d$, $y=(a^2-d^2)/4d$.

For let $y=x+n$, whence $x^2+ax=x^2+2nx+n^2$, or $x=n^2/(a-2n)$.

Now x is integral if, and only if, $a-2n$ is a factor of n; that is, if an integer k exists such that $n=kd$ [denoting $a-2n$ by d]. But then $a=d+2n=d(1+2k)$; so that integral values of x exist if, and only if, d is contained in a an odd number of times.

Using any such number d we have, $k=(a-d)/2d$, $n=\frac{1}{2}(a-d)$, $x=(a-d)^2/4d$, $y=x+\frac{1}{2}(a-d)=(a^2-d^2)/4d$, as stated.

Remarks. (I) Negative values of d are admissible, as the factors x and $x+a$ merely have their numerical values interchanged.

(II) If a be prime, $d=\pm1$ gives the only solution except the trivial case $d=a$, $x=y=0$.

(III) Numerical examples follow:

$a=15,\ d=1,\ x=49,\ y=56.$
$d=3,\ x=12,\ y=18.$
$d=5,\ x=5,\ y=10.$
$a=18,\ d=2,\ x=32,\ y=40.$
$d=6,\ x=6,\ y=12.$

II. Solution by A. H. HOLMES, Brunswick, Me.

Solving with respect to x, $x=\dfrac{\pm\sqrt{(4y^2+a^2)}-a}{2}$.

Take $a=p^2-q^2$ and $y=pq$.

Then $x=\dfrac{\pm(p^2+q^2)-p^2+q^2}{2}=q^2$ or $-p^2$, in which p and q can be any integers, $p>q$.

Put $q=1$, $p=2$. Then $x=1$ or -4, $a=3$, and $y=2$.

Secondly. Put $a=$any integer, say 23. $x^2+23x=y^2$.

$\therefore x=\dfrac{\pm\sqrt{(4y^2+529)}-23}{2}$.

Put $y=pq$ and $23=p^2-q^2$.

Then $x=\dfrac{\pm(p^2+q^2)-p^2+q^2}{2}=q^2$ or $-p^2$.

$p^2-q^2=(p+q)(p-q)=23=23\times1$.

$\therefore p+q=23$, and $p-q=1$. $\therefore p=12$ and $q=11$.

$\therefore x=121$ or -144, $y=132$.

Also solved by the late G. B. M. Zerr, and S. Lefschetz.

PROBLEMS FOR SOLUTION.

ALGEBRA.

349. Proposed by JOSEPH A. NYBERG, Student, University of Chicago.

To show that the determinant of the nth order:

$$D_n=\begin{vmatrix} C & -1 & 0 & 0 & 0 & 0 & 0 & . & . \\ -1 & C & -1 & 0 & 0 & 0 & 0 & . & . \\ 0 & -1 & C & -1 & 0 & 0 & 0 & . & . \\ 0 & 0 & -1 & C & -1 & 0 & 0 & . & . \\ 0 & 0 & 0 & -1 & C & -1 & 0 & . & . \\ 0 & 0 & 0 & 0 & -1 & C & -1 & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}$$

has the value: $D_n=C^n+\sum_{r=1}^{n}(-1)^r\dfrac{(n-r)(n-r-1)\ldots(n-2r+1)}{r!}C^{n-2r}$.

350. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations: $x+y+z=a_0$,
$x+yu+zv=a_1$,
$x+yu^2+zv^2=a_2$,
$x+yu^3+zv^3=a_3$,
$x+yu^4+zv^4=a_4$.

CALCULUS.

304. Proposed by H. C. FEEMSTER, York College, York, Neb.

Reduce $axyp^2+(x^2-ay^2-b)p-xy=0$ to Clairaut's form, and hence solve the equation.

305. Proposed by C. N. SCHMALL, New York City.

Prove $\int_\beta^x \frac{dx}{\sqrt{[(\alpha-x)(x-\beta)]}} = 2\cos^{-1}\sqrt{\frac{\alpha-x}{\alpha-\beta}}$. [Edwards' *Integral Calculus for Beginners*, p. 84, ex. 4.] Does this result hold when the upper limit is changed from x to α?

MECHANICS.

358. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two heavy particles connected by a string, length l, lie one on each of two inclined planes with common horizontal edge and of angles α and β. The inclination of the string to the edge varies as the inclination to the horizon of a simple pendulum of length $l(\sin\alpha+\sin\beta)$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

178. Proposed by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

Find a formula which gives all the integral solutions prime to 5 of the congruence $y^2+z^2\equiv 0 \pmod{5^4}$.

179. Proposed by V. M. SPUNAR, Chicago, Ill.

Solve the equation in integers, $x^n+y^n+z^n+xyz=100x+10y+z$.

180. Proposed by A. H. HOLMES, Brunswick, Maine.

Find integral values for x and y in the following: $96x-96y+21=\square$.

NOTES AND NEWS.

Dr. Arnold Emch, of Basel, Switzerland, has sailed for America and expects to assume his duties as Assistant Professor of Mathematics in the University of Illinois at the beginning of the second semister. Professor Emch is the author of numerous articles, which appeared in various journals

of Europe and America. In 1905 he published a book entitled "An Introduction to the Projective Geometry and its Applications," which has been favorably reviewed by Karl Doehlemann in *Zeitschrift für Mathematik und Physik*, Vol. 58 (1908), page 332; and also by Lothar Heffter in *Archiv der Mathematik und Physik*, Vol. 13 (1908), page 72. This book as well as the articles mentioned above deal, to a large extent, with the applications of mathematics to the problems of the engineer and the physicist. M.

The next regular meeting of the American Association for the Advancement of Science will be held at Washington, D. C., under the Presidency of Professor C. E. Bessey, University of Nebraska. At this meeting Professor E. H. Moore, University of Chicago, is expected to give his address as retiring Vice-President. Professor E. B. Frost, Director of Yerkes Observatory, was elected Vice-President and Chairman of Section A, during the recent Minneapolis Meeting of the Association. M.

A joint meeting of mathematicians and engineers was held during the holidays at Minneapolis, Minnesota, in connection with the convocation of the American Association for the Advancement of Science, to hear the report of the committee of twenty on the teaching of mathematics to students of engineering. Three years ago a similar meeting was held in Chicago through the Chicago Section of the American Mathematical Society. The Chicago meeting aroused much interest and resulted in the appointment of a committee of twenty, under the chairmanship of Professor E. V. Huntington, of Harvard University, to consider the whole question of the teaching of mathematics to students of engineering in this country, and to report its recommendations to the Society for the Promotion of Engineering Education at its summer meeting to be held at Madison, Wisconsin, in June, 1910.

In the early part of its investigation the committee collected a large amount of information in regard to the present status of mathematical instruction for engineering students. Since that time, however, a much more inclusive inquiry has been undertaken by the International Commission on the Teaching of Mathematics. In order to avoid unnecessary duplication, this committee voted to turn over all the results of its own inquiry in this field to the larger commission, to be worked up in accordance with the general scheme adopted by that commission, and to be incorporated in their report. This material is therefore not included in the present report.

Aside from the collection of data, Professor Huntington's committee decided that the most important need at the present time is a series of synoptical text-books, which shall present: (1) the fundamental principles of the science, in compact form, and (2) a classified and graded collection of problems (which would naturally be subject to continual change and expansion). It is their hope that the Minneapolis report, which is confined to the first part of the desired text-book, will stimulate throughout the country practical contributions toward the second. S.

Copies of these syllabi were distributed among the hundred or more engineers, physicists, and mathematicians who joined in a good fellowship dinner on the evening preceding the presentation of the report. Additional copies may be had upon request from the chairman. The following quotations from the preface will indicate its scope and purpose:

"The object of this report is to present a synopsis of those fundamental principles and methods of mathamatics which, in the opinion of the committee, should constitute the minimum mathematical equipment of the student of engineering.

"It is hoped that this report may be serviceable in two ways: first, to the teacher, as an indication of where the emphasis should be laid; and secondly, to the student, as a syllabus of facts and methods which are to be his working tools. It does not include data for which the student would properly refer to an engineer's hand-book; it includes rather just those things for which he ought never to be obliged to refer to any book—the things which he should have constantly at his fingers' ends.

"The teacher of mathematics should see to it that at least these facts are perfectly familiar to all his students, so that the professor of engineering may presuppose, with confidence, at least this much mathematical knowledge on the part of his students. On the other hand, if the professor of engineering needs to use, at any point, more advanced mathematical methods than those here mentioned, he should be careful to explain them to his class.

"The defects in the mathematical training of the student of engineering appear to be largely in the knowledge and grasp of fundamental principles, and the constant effort of the teacher should be to ground the student thoroughly in these fundamentals, which are too often lost sight of in a mass of details.

"The order in which these topics should be taken up must be left largely to the discretion of the individual teacher. The committee is firmly of the opinion, however, that whatever order is adopted, the principal part of the course should be problems worked by the students, and that all these problems should be solved on the basis of a small number of fundamental principles and methods, such as are here suggested."

The report was freely discussed and frankly criticised by a large number of speakers, and the discussion finally led to the following resolution, which was unanimously adopted:

Resolved, That this body tenders hearty thanks to Professor Huntington for the great interest which he has shown in this work and the untiring service which he has given to it; that we commend the work of the entire committee for the preparation of a report which it is believed must operate for betterment along the lines of its recommendations; and while not prepared to approve in all respects the details, especially in the syllabus on dynamics, as shown by the full and free discussion at this meeting, yet we

heartily endorse the spirit of the report and thank the officers of the Society for the Promotion of Engineering Education, who have shown their friendly coöperation in offering to publish these syllabi in the official *Bulletin*, for the purpose of drawing out further criticisms and suggestions either in printed papers or in written communications to the chairman of the committee.

The further report of this committee is to be presented at the next meeting of the Society for the Promotion of Engineering Education in June, 1911. The present meeting was organized, as was the original meeting in Chicago in 1907, by the Chicago Section of the American Mathematical Society.

BOOKS AND PERIODICALS.

Engineering Descriptive Geometry. A Treatise on Descriptive Geometry as the Basis of Mechanical Drawing, Explaining Geometrically the Operations Customary in the Draughting Room. By F. W. Bartlett, Commander United States Navy, Head of Department of Marine Engineering and Naval Construction at the United States Naval Academy, and Theodore W. Johnson, A. B., M. E., Professor of Mechanical Drawing, United States Naval Academy, Member of American Society of Mechanical Engineers. 8vo. Cloth, vi+159 pages. New York: John Wiley & Sons.

This work follows the plans of practical draughtsmen, in that it aims to make the subject of practical use rather than a subject for the mathematical student. Unlike the older works on the subject, but like the more modern texts, the object to be delineated is placed behind the vertical plane and below the horizontal plane. Use is also made of side planes. The book will be found useful in engineering courses. F.

Elements of Plane Trigonometry. A Text-Book for High Schools, Technical Schools, and Colleges. By Robert E. Moritz, Ph. D. (Neb), Ph. N. D. (Strassburg), Professor of Mathematics, University of Washington. First Edition, First Thousand. 8vo. Cloth, xvi+359 pages+91 pages of Tables. Price, $2.00. New York: John Wiley & Sons.

The author believes that Trigonometry is college mathematics *par excellence* and therefore in this text aims to present the science in as simple and as attractive manner as possible. He aims to make the subject less technical, by introducing considerable historical matter, and by assuming that the student is unfamiliar with much of college algebra, and knows very little of the rudiments of mathematics. The author does not hesitate in the advanced part of the subject, to lay down the fundamental notions on which the whole structure rests. Thus in dealing with the Theorems of Demoivre and Euler no knowledge of imaginaries is assumed. The Hyperbolic Functions are treated in a seperate chapter. Here, too, the analogies between them and the Circular Functions are pointed out.

The book is full of interesting material, and will be found very serviceable in the class-room. F.

The Monist. A Quarterly Magazine Devoted to the Philosophy of Science. Edited by Paul Carus. Price, $2.00 per year. The Open Court Publishing Co.

The January number contains, among other articles, an interesting article on Schopenhauer's Type of Idealism, by W. M. Salter; one on Professor Mach and his work, by Paul Carus; and one on From Protagoras to Williams James, by John E. Boodin. Criticisms On the Magic Circle, by Herbert Chatley; Notes on Pandiagonal and Associated Magic Squares, by L. S. Frierson; Two more forms of Magic Squares, by A. J. Edmunds.

The Physical Review. A Journal of Experimental and Theoretical Physics, conducted with the coöperation of the American Physical Society. By Edward L. Nichols and others. Price, $5.00 per year. Ithaca, N. Y.

The January number contains a number of valuable articles, among which we mention, Volume Ionization Produced by Light of Extremely Short Wave-length, by Fredric Palmer; Studies in Luminescence, by Edward L. Nichols and Ernest Merritt; Experiments in Impact Excitation, by George Nasmyth.

The American Journal of Mathematics. Edited by Frank Morley with the coöperation of other mathematicians. Price, $5.00 per year in advance.

The January number contains the following articles: On a Class of Cubic Surfaces with Curves of the Same Species, by John Eisland; The Automorphic Transformations of the Binary Quartic, by A. H. Wilson; Theorems on Simple Finite Polygon and Polyhedron, by N. J. Lennes; On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients, by F. R. Moulton and W. D. MacMillan.

Technical World Magazine. Price, $1.50 per year. Chicago, Illinois.

The January number contains an article on Bringing the Dead to Life, and many other articles that will interest the scientifically inclined.

Current Literature. A Review of the Times. Edited by Edward J. Wheeler. Price, $3.00 per year.

The leading articles of the January number are, After the Political Cyclone; The Wonder of Mrs. Eddy; America's Medical Hell; The Dynastic Dream in China; and Our Century's Greatest Rebel.

McClure's Magazine. An International Monthly Magazine. Published by S. S. McClure, New York City. Price, $1.50.

One of the articles of great importance in the February number is The Mormon Revival of Polygamy, by B. J. Hendrick.

School Science and Mathematics. A Journal of Science and Mathematics for Teachers in Secondary Schools. Price, $2.00 per year. Chicago, Illinois.

Any teacher of mathematics or science who does not read this or some journal devoted to his special subject, should be disqualified to teach. Only a few years ago the number of teachers of this class was legion. F.

NOTE.—Dr. G. E. Whalen, of the University of Illinois, Urbana, Illinois, will in future edit the Number Theory Department in the MONTHLY.

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THE POLES OF FINITE GROUPS OF FRACTIONAL LINEAR SUBSTITUTIONS IN THE COMPLEX PLANE.

By W. B. CARVER, Cornell University.

In his Lectures on the Icosahedron, Klein shows that any finite group of fractional linear substitutions may be transformed to a canonical form such that each substitution will correspond (by the ordinary stereographic projection) to a rotation of the sphere which has the unit circle as a great circle. In this canonic form the two poles of each substitution are what we may call *skew-inverse* with respect to the unit circle; *i. e.* they lie on a line with the center, and the product of their distances from the center is -1. Two points may be said to be *skew-inverse* with respect to any circle C of radius r if they are collinear with the center and the product of their distances from the center is $-r^2$. This relation, unlike the ordinary *inverse* relation, is not invariant under the general fractional linear substitution. Hence the following two questions (which are practically equivalent) naturally arise:

(1). If a finite group of fractional linear substitutions is not in canonic form, do its operations correspond by a stereographic projection to rotations of a sphere having some other circle of the complex plane as a great circle?

(2). If upon all the pairs of points skew-inverse with respect to some circle of the complex plane one operates with an arbitrary fractional linear substitution, will they go into pairs skew-inverse with respect to a circle; and if so, what circle?

It is the purpose of this paper to answer these questions in reverse order.

If the points of a sphere are projected upon the plane of a great circle C from a pole of C, a familiar one-to-one correspondence is established which has the following well-known properties:

Circles on the sphere correspond to circles* in the plane.

Great circles on the sphere correspond to circles in the plane which

* The term circle is to be understood to include straight lines.